Optimal design of plane elastic membranes and application to 3D form-finding

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Recently, in the paper [1], a novel problem of optimal design of plane membranes was put forth. Therein, one varies the membrane's pre-stress tensor field, which puts the formulation in the class of plastic design methods. The present talk virtually brings this idea to elastic membranes modelled by Föppl's theory [4], where the in-plane strain is expressed via the operator known from von Kármán's plate theory: the strain is linear with respect to the in-plane deformation and non-linear with respect to the membrane's deflection. Our goal is to find a distribution of the membrane's thickness that – under the constraint on the volume – minimizes the compliance for the out-of-plane load. This convex design problem proves to be well-posed, and it gives rise to a pair of mutually dual convex variational problems of the form that was studied in [1]: the stressed based problem (P) and the displacement based problem (P*). From the mathematical perspective this pair is reminiscent of the one known from the Michell theory, or from the Free Material Design problem [3]. The proposed design setting is general enough to consider a whole class of non-linear constitutive laws that rules out compression of the membrane. The numerical strategy for the optimal membrane problem consists in discretizing the pair (P), (P*). This is achieved via the finite element method that is newly developed and tailored specifically for this pair.

The paper [2] showed that the optimal design of 2D plastic membranes paves a way to optimal 3D form-finding, which allowed to efficiently identify precise plastic grid-like approximations of optimal vaults. In this talk we follow this approach for the optimal elastic plane membranes. We prove that the surface on which the optimal form should concentrate coincides with the deformed optimal membrane, and one recovers the material distribution through unprojecting the membrane's thickness. Effectively, the foregoing finite element method furnishes suboptimal shells that approximate the optimal 3D form.

References

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